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On the Generation of Surfaces by the Motion of Plane Curves.

Dissertation
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Introduction

1. I shall first mention a few properties of homothetic curves and then state the object of this investigation. Rectangular coordinates will be used throughout the work. Suppose

$$f(x, y) = 0$$

the equation of a curve C. The equation

$$f[kx + (1-k)x_0, ky + (1-k)y_0] = 0,$$

involving the parameter k, represents all curves homothetic to C, x_0, y_0 being the coordinates of the homothetic centre. The parameter k is called the homothetic ratio. If the origin is taken as the homothetic centre the equation of the homothetic curves reduces to

$$f(kx, ky) = 0.$$

Corresponding to a given homothetic centre there is a system of curves homothetic to C which leaving out of account their

position in the plane represent all the curves homothetic to c. The homothetic figure of a right line is a parallel right line. The angle of two lines is equal to that of their homologous lines. The tangents at homologous points of homothetic curves are parallel. To pass from the curve

$$f(x, y) = 0,$$

to a homothetic curve we must make the substitution

$$x = \kappa x' + (1-\kappa) x_0,$$

$$y = \kappa y' + (1-\kappa) y_0,$$

where x_0, y_0 are the coordinates of the homothetic centre.

2. In this paper I shall apply a certain method, which depends upon the properties of a system of homothetic curves, to the generation of surfaces, and shall show how certain properties of surfaces may

[illegible]

A surface may be defined by a property common to each of its points. In this case we obtain the equation of the surface by translating this property analytically.

Or, more usually a surface is defined by the movement of a line in space. [illegible]

$$f(x, y, z, a) = 0, \quad F(x, y, z, a) = 0$$

the equations of a line containing an arbitrary parameter a; if a varies in a [illegible]

[illegible]

equation of the surface is obtained by eliminating a between the equations of the moving line. In the method which I have used a surface is defined by the motion of a variable plane curve whose plane always passes through a fixed right line. As the plane rotates about [illegible]

line as an axis, the curve changing its form and its position in the plane. It is generally more convenient to define the motion of this plane only by subjecting it to the condition of moving on certain fixed curves which may be either plane or twisted. In each position of the generating curve [illegible] to define it by means of an equation between its rectangular coordinates (x, y).

Any surface whatever can be [illegible] in this way [illegible]

[illegible]

It will be shown that if the axis of rotation is taken as the axis of z and the generating curve is a straight line, the equation of the surface will be of the form

$$y = x\varphi(z) + \psi(z),$$

[illegible] of the generating curve [illegible] such the equation of the surface will be of the form

$$z^2 + z\phi_1\left(\frac{y}{x}\right) + x^2 + y^2 + x\phi_2\left(\frac{y}{x}\right) + \phi_3\left(\frac{y}{x}\right) = 0.$$

[illegible] of the [illegible] and that in the case of surfaces of revolution.

A surface of revolution is generated by the rotation of any curve about a fixed straight line in its plane. During the rotation the [illegible] points of the plane curve [illegible] the straight line about which the rotation takes place is called the axis of the surface. Each point of the curve describes a circle [illegible] perpendicular to the axis [illegible] generating curve is called a meridian of the surface of revolution. In order to find the equation

of a surface of revolution. [illegible] Take for the axis of rotation the axis of y, and for the plane in which the generating curve was originally placed the xy plane, of a system of rectangular coordinates. Suppose that the equation of the generating curve is

$$y = f x.$$

[illegible] the [illegible] any plane which passes through the axis of y, and [illegible] rectangular coordinates [illegible] these [illegible] must cut out from the surface one of the original congruent curves, the equation of [illegible]

$$y = f(r)$$

[illegible] dimensions [illegible]

$$f [illegible] x^2 + z^2$$

[illegible]

$z = f(x^2+y^2)$

is the equation of the surface of revolution. In this case the equation of the plane curve is given by means of the coordinates x^2+y^2 as the curve is invariable if we change the projection on any [illegible] of the plane [illegible] [illegible] the function, so the only one of the [illegible] [illegible]. The surface

4. The [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] faces of revolution depends upon a system of homothetic [illegible] circles, the axis of the surface being perpendicular to the plane of these circles and passing through the homothetic centre.

5. The same method can be applied

to the study of [illegible] curves.
If two plane curves lie in the
same plane and are referred to the
coordinates, their intersections [illegible]
found by the [illegible] methods, and
[illegible]

I.

1. Being given any system of homothetic curves in the plane of xy, it may, on taking the origin at the homothetic centre, be represented by the equation

(1) $$f(Kx, Ky) = 0$$

The equation

$$f(x, y) = 0$$

represents a curve of the system which will be called the fundamental curve, and the curve

$$f(Kx, Ky) = 0$$

will be referred to as the curve K.

If we denote by ρ the absolute value of the distance from the origin to a point P of the fundamental curve and by ρ' the [illegible]

[illegible]

[illegible]

P' of the curve (h) we shall have

(3) $$h = \frac{c}{t}.$$

[illegible]

These curves a perpendicular is erected to the plane of xy; the extremities of these perpendiculars trace out a surface [illegible]

curve (h) by

(4) $$\phi(x, y, z) = 0$$

the equation of the surface will be obtained by eliminating h between [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

(5) $$\psi\left(x, \frac{y}{t}, \frac{y}{x}\right) = 0,$$

[illegible]

[illegible]

[illegible]

of the section of the surface by the plane ZOT. If

$$f(x, t) = 0$$

is an algebraic curve, of the n^{th} order, say, then, if for any line through the origin we get n finite [illegible] values of t, [illegible] [illegible] each of these values will correspond [illegible] [illegible] accordingly, the section of the surface by a plane through the axis OZ will consist of the n plane curves

(1) [illegible]

[illegible] [illegible] [illegible] [illegible] [illegible] directions [illegible] [illegible] [illegible] of the finite [illegible] [illegible] OT, [illegible] the positive direction for t and [illegible] [illegible] [illegible] $O'T$; OT' is the positive direction. The surface obtained by eliminating k between

$$f(kx, ky) = 0$$

[illegible]

is therefore generated by the motion of these in plane curves as their common plane rotates round the axis of y.

The equation

$$\phi(y, \ldots) = 0$$ [illegible]

may be called the type equation of the generating plane curves [illegible] of the [illegible] contained at an ordinary point of the fundamental curve. [illegible] only [illegible] the equation of the [illegible] is of the [illegible] n^{th} degree and the section of the surface [illegible] for the [illegible] the special position of the cutting plane when '$T\sigma T'$' is tangent to the fundamental curve with [illegible]

$\phi($[illegible]$)$ [illegible]

and the curve

$\psi($[illegible]$) = 0$, [illegible]

When '$T\sigma T$' is tangent [illegible]

curve at the origin, the sections are composed of

$$\psi\left(z, \frac{t}{l_i}, \frac{y}{x}\right) = 0, \; l_i \neq 0, \; (i = 3, 4, \ldots n).$$

[illegible]

If the homothetic centre is at an m-tuple point the section will consist of the curves:

[illegible]

[illegible]

When TOT' is tangent to the fundamental curve at the origin [illegible]

$$\psi\left(z, \frac{t}{l_i}, \frac{y}{x}\right) = 0, \; l_i \neq 0, \; (i = m+2, m+3, \ldots n),$$

[illegible] $m+1$).

2. If we desire the line around which the [illegible] and which cuts the plane of xy [illegible] to be parallel to the axis of y, [illegible] as follows:

$$f\left[Kx + (1-K)x_0, \; Ky + (1-K)y_0\right] = 0,$$

$$\psi\left(z, \frac{1}{K}, \frac{y - y_0}{x - x_0}\right) [illegible] 0,$$

x_0, y_0 being the coordinates of the [illegible]

2. Let us now consider the circle

$$x^2 + y^2 = \rho^2$$

as the fundamental curve.

The system of homothetic curves will be

$$h^2 x^2 + k^2 y^2 = \rho^2,$$

where we [illegible]

$$y^2 = \frac{x^2 [illegible]}{\rho^2}$$

Then from the differential equation

$$\varphi(x, y, p) = 0$$

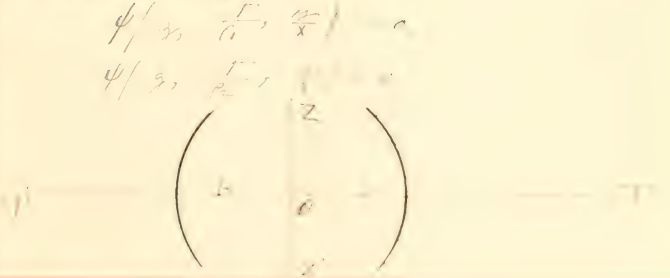
[illegible] the fundamental circle [illegible] with respect to the origin O, where

$$r = h$$

we have the equations

$$\psi_1\left(x, \frac{r}{\rho_1}, \frac{y}{x}\right) = 0,$$

$$\psi\left(y, \frac{r}{\rho_2}, [illegible]\right) = 0.$$



a point in some curve lying in the plane ZOT which are symmetrical with respect to the axis of Z and then to consider a point the section [illegible] of the plane ZOT from the surface obtained by eliminating [illegible] between

$$\psi(z, t, r) = 0,$$

$$r = \sqrt{x^2+y^2}.$$

It is evident that the section of the surface by any plane through the axis of Z is symmetrical with respect to that axis. If [illegible] is actually known [illegible] the equations of the [illegible] the section coinciding with the [illegible] plane

The equation of the surface will then have the form

$$\psi\left(z, \sqrt{x^2+y^2}, \frac{y}{x}\right) = 0,$$

or more generally

[illegible]

If the surface is a surface of revolution, the section does not change with the cutting plane. This is indicated by the absence of [illegible] from the typical equation, and the equation of the surface becomes

$$\psi\left(\sqrt{x^2+y^2},\, z\right) = 0.$$

4. Since the circle is a curve of the second order it would seem that, according to the general theorem given above, the number of [illegible] curves would be two, and yet it is evident that there are not always two distinct curves, as for instance in the rotation of a curve about a diameter such as the axis of y. In fact there [illegible] two curves, but they may be [illegible] by the typical equation

[illegible]

[illegible]

ponding tangent will be constant and this generating curve will be a parallel inclined to the axis of z.

5. Knowing that the equations of the sections of any proposed surface by planes through the axis of z can be represented by one and the same equation involving an arbitrary parameter determining the position of the section, there is no loss of generality in writing this equation

$$\varphi\left(z,\ t,\ \frac{y}{x}\right) = 0,$$

since $\frac{y}{x}$ can be taken as that parameter. Accordingly by what precedes, any surface whatever whose sections by planes through the axis of z are symmetrical with respect to this axis, can be represented in the foregoing manner by means of an equation of [illegible] and therefore the general [illegible] surfaces of this class.

$\frac{d}{dt}\left([illegible] \sqrt{x^2+y^2} - x \right) = 0.$

In the case of a surface in which the sections are not symmetrical with respect to the axis of y, the same method would not readily be applicable, [illegible] [illegible] symmetrical with respect to the axis of y. [illegible] [illegible] the value of t, it may be brought [illegible] through the [illegible]. [illegible] of the [illegible] ation may take place through cancellation of the radical and without raising to powers. [illegible] must be understood in the value of $t = \sqrt{x^2+y^2}$ one surface will be represented when one sign is taken, the symmetrical surface when the other sign is taken.

6. As an example of the [illegible] [illegible] surface [illegible] [illegible] the plane

$$x + y + z - 1 = 0$$

[illegible] straight line which passes through the point (0, 0, 1) on the axis of z. For the type equation of the generating line let us take

$$z - 1 = m x$$

[illegible] the tangent of the angle which the line makes with the [illegible]. [illegible] suppose [illegible] the function of the [illegible] and accordingly to be a function of x and y, we shall suppose

$$m = \frac{x + y}{x^2 + y^2}$$

[illegible] shall [illegible] type equation [illegible]

$$z - 1 = \frac{x + m}{x^2 + y^2} \cdot x.$$

To get the [illegible] equation [illegible]

substitute for t its value $\sqrt{x^2+y^2}$, and [illegible]

$$x+y+z-1=0,$$

$$z-1-x-y=0.$$

7. To generate a surface of revolution we must eliminate t between the equation of the type curve

$$\psi(z, t)=0,$$

and $\quad t^2 = x^2+y^2.$

If only even powers of t occur in the equation of the type curve, the two generating curves will coincide, and only one [illegible] with respect to the axis of z. If odd powers of t occur there is nothing to make the irrationality disappear without squaring [illegible] for the type equation does not contain $\frac{y}{x}$. Hence therefore [illegible] by invariable curves [illegible]

a perfect one. The points [illegible] change the[illegible] to the same surface.

8. In the case of homothetic concentric circles we have just seen that if $\frac{y}{x}$ does not occur in the type equation, the section of the surface does not change with the cutting plane. This is the only elementary system which possesses this property, for in all other systems [illegible] change with the cutting plane. The following are some examples in which advantage is taken of the variation of [illegible]. [illegible] take the circle

$$x^2+y^2=2x,$$

as the fundamental curve. We obtain

$$t = \frac{x^2+y^2}{x}.$$

The type equation [illegible]

[illegible]

[illegible]

[illegible] made by

$$t^2 = 4c^2y \qquad [illegible]$$

$t^2 = 4c^2y$, [illegible] $x^2 = 0$.

The section therefore consists of the parabola

$$t^2 = 4c^2z,$$

and the axis of z counted twice. The distance of the focus of the parabola from the origin is equal to the square of the distance from the origin to the circle

$$x^2 + y^2 = 2x,$$

measured along the radius vector OT. Since the section consists of a parabola and two [illegible] lines, the surface is of the fourth order. The equation of ~~the~~ section of the surface by the plane $z = 0$ is $t^2 = 4c^2y$, [illegible] which represents the axis of y counted [illegible] times. The axis of y is therefore a double line of the surface.

surface has a triple point at the origin, for any line drawn through the origin and lying in the plane ZOT meets the parabola

$$t^2 = 4l^2 z;$$

once in point at the origin and the axis of z in two coincident points. Any line therefore, drawn through the origin meets the surface in at least three coincident points that is the origin is a triple point. The line TOT' is tangent to the parabola

$$t^2 = 4l^2 z$$

at the origin and meets the axis of z in two coincident points, whence it meets the surface in four coincident points. The plane of xy therefore forms part of the cone of tangents to the surface at the origin. Since the section of the plane of yz consists of the axis of z and two lines through the origin [illegible]

the origin and lying in this plane touches the surface in four coincident points, hence this plane forms part of the cone of tangents. It is readily seen that this plane is a stationary tangent plane and therefore the cone of tangents consists of the plane of xy and of the plane of yz counted twice. Eliminating t between

$$t^2 = 4\rho^2 z,$$

$$\frac{t}{\rho} = \frac{x^2+y^2}{2x},$$

we get for the equation of the surface

$$(x^2+y^2)^2 - 16x^2 z = 0.$$

The sections of the surface by planes parallel to the plane of xy are compounded of circles. For example the section by the plane $z = 4$ [illegible] the circles

$$(x^2+y^2)^2 = 64x^2$$

$$x^2+y^2 = \pm 8x$$

of the 4, one of these projections is the fundamental circle

$$x^2 + y^2 = rx,$$

while the other projects into the symmetric of the fundamental circle with respect to the origin.

The surface may therefore be generated by the motion of a circle whose plane is parallel ~~to~~ to the plane of xy, the extremities of a diameter of the circle moving upon the axis of z and upon the generating parabola in the plane of xz as shown in the figure.

The centre H of the generating circle moves on the parabola $x^2 = z$.

Suppose now that the equation of the type curve is

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1.$$

The section consists of the ellipse

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1,$$

and the axis of z counted twice.

We are to form the equation of the surface consisting of the axis of z counted twice and of the sheet generated by the motion of an ellipse whose plane

[illegible]

one axis of the ellipse being constant and the extremity of the other moving on the circle

$$x^2 + y^2 = ax.$$

Eliminating β from the equation of the ellipse by means of

$$\beta = \frac{x^2+y^2}{a}$$

we get for the equation of the surface

$$4x^2y^2 - c^2(x^2+y^2)^2 - 4c^2x^2 = 0,$$

which represents a quartic surface with a double point at the origin. If the generating curve is the hyperbola

$$\frac{y^2}{c^2} - \frac{x^2}{\beta^2} = 1,$$

the equation of the surface is

$$4x^2y^2 - c^2(x^2+y^2)^2 - 4c^2x^2 = 0.$$

The equation of the asymptotes of the generating hyperbola is

$$\frac{x^2}{c^2} - \frac{y^2}{\beta^2} = 0,$$

whence the asymptotes of ... surface

generated by these asymptotes are

$$4x^2y^2 - c^2(x^2+y^2)^2 = 0,$$

which breaks up into the cones

$$2xy - c(x^2+y^2) = 0,$$

$$2xy + c(x^2+y^2) = 0,$$

and each of these cones is asymptotic to the surface. The cones and the surface pass through the axis of z for a common tangent plane, the plane of yz, the generating hyperbola is drawn to these axes.

Suppose that the hyperequation is

$$(y^2x-c^2)^2 = (x-c)^5.$$

The point $x = c$, $y = 0$ is evidently a cusp, at which $x - c = 0$ is the tangent, as in the accompanying figure.

The type equation represents the curves

$$\rho^3(t-\rho z)^2-(t-\rho)^5=0,\quad \rho \neq 0,$$

and $$\rho^3(t-\rho z)^2-(t-\rho)^5=0,\quad \rho=0,$$

that is the section consists of the curve

$$\rho^3(t-\rho z)^2-(t-\rho)^5=0,$$

and of the axis of y counted five times. The surface will be of the tenth order and will have a quintuple point at the origin. The section in the plane of y z will consist of the axis of y counted ten times, and this axis is a quintuple line of the surface. The locus of the point P will be a cuspidal line of the surface and its equations are

$$y=1,\quad x^2+y^2=2x$$

The tangent to the cusp will generate the cone $$x^2+y^2=2xy.$$

Eliminating ... between the equations

[illegible]

$$\frac{t}{\rho} = \frac{x^2+y^2}{2x},$$

we get for the equation of the surface

$$8x^3(x^2+y^2-2xz)^2-(x^2+y^2-2x)^3=0.$$

Suppose that the straight line

$$ax+by=1,$$

is the fundamental curve. We obtain

$$\frac{t}{\rho} = ax+by,$$

and there is but one generating curve, since the straight line is a curve of the first order. Suppose that the type generating

$$y - t = c.$$

The problem is to form the equation of the surface generated by the motion of a straight line which always passes through the origin and the tangent of whose angle with the axis of y is numerically equal to the distance from the origin to the line.

measured along the line TOT'. We get for the surface the plane

$$ax + by - z = 0$$

If the type equation is

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1,$$

the surface will be generated by the motion of an ellipse having one axis constant while the extremity of the other moves on the line

$$ax + by = 1.$$

We get for the equation of the surface

$$\frac{z^2}{c^2} + (ax+by)^2 = 1,$$

~~which by an easy change of the axes becomes~~

$$\frac{Z^2}{c^2} + X^2 = 1,$$

which represents an elliptic cylinder.
With this system we can only represent planes and cylinders, [illegible] [illegible] not appear in the type equation. If

$$\varphi(z, t) = 0$$

in the type equation, the equation of the surface will be

$$\psi(y,\ mx + l z) = 0,$$

which is the type of the equations of cylindrical [illegible] surfaces [illegible]. Suppose that the base of [illegible]

$$k^{n+m} x^m y^n = 1,$$

and that the sections of the surface are given by the type equation

$$z^k = c z^{m+n}.$$

We readily get for the equation of the surface

$$x^m y^n z^k = c,$$

which represents a class of surfaces whose bases of sections can be obtained. Now let us take some special cases of these surfaces. Suppose that the fundamental curve is the equilateral hyperbola

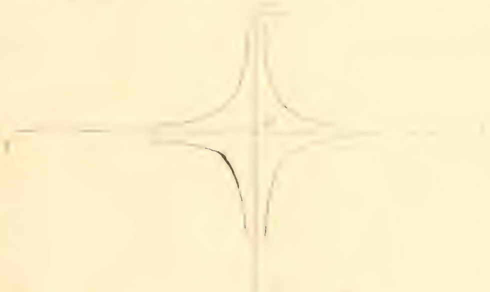
[illegible]

$$t = \sqrt{x \cdot y}.$$

For the type equation let us take

$$t = T.$$

Here $t = T$ and the type equation represents the equilateral hyperbolas which are symmetrical with respect to the origin. The section of the surface by a plane through the axis of y is as shown in the figure.



If the semi-axes of these hyperbolas are represented by a, we have

$$a = \sqrt{2}\,t,$$

where t denotes the distance measured from the origin to the fundamental hyperbola.

$$xy = 1,$$

along the radius vector TOT'. The surface is asymptotic to the axis of y and also to the plane of xy at infinity, and the equation is

$$z^2 x y = 1.$$

Suppose that the parabola

$$y^2 = x,$$

is the fundamental curve and that the type equation is

$$2y \cdot T = \rho.$$

This type equation stands for the equilateral hyperbola

$$xyT = \rho$$

and

$$xy\rho = 1, \quad (\rho = 0).$$

The last equation represents the cone of the surface with half the third axis and the origin will be at an ordinary point of the surface. The cross section of the generating hyperbola is perpendicular to the axis

from the origin to the fundamental parabola,

$z^2 = 2x$,

are applied in the direction of T. The section of the surface are shown in the figure.

The section, as due to the axes of y and y' in the plane of yz, form in this plane the equilateral hyperbola referred to these lines. The surface consists of two sheets, one of which lies above the plane of xy and on that side of the plane of yz which is the positive direction of x, the other lies below the plane of xy and on that side of the plane of yz which is the negative direction of x. The two sheets are joined together along the axes of y and y'.

which are lines of the surface. The equation of the surface is

$$xy - z = 0.$$

9. The quadric surfaces are readily generated in the following way. Let the type equation be

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = f(x)$$

and the fundamental curve be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We get for the equations of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = f(x),$$

which represents a surface all of whose [illegible] through the [illegible] of [illegible]. If $f(x) = 1$ the surface is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The type equation shows that it is generated by the motion of an ellipse one of whose axes remains equal to a while the velocity of the other varies in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} =$$

Let the type equation be

$$\frac{t^2}{\rho^2} - \frac{z^2}{c^2} = 1,$$

while the fundamental curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation of the [illegible] is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

[illegible]
[illegible]
[illegible]

one of whose semiaxes is equal to c while the extremity of the other moves on the [illegible]

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Let the type curve be

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1,$$

and take the same fundamental curve as before. The equation of the corresponding surface is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

[illegible]

The type equation shows that it was generated by the motion of an ellipse, one of whose semi-axes is equal to z while the extremity of the other moves on the hyperbola

$$\frac{x}{a} - \frac{y^2}{b^2} = 1.$$

If the type equation be

$$\frac{x}{c} = z,$$

and the fundamental curve be

$$\frac{x}{a} + \frac{y^2}{b^2} = 1$$

the equation of the corresponding surface is

$$\frac{x}{a} + \frac{y^2}{b^2} = z,$$

which represents an elliptic paraboloid.

Using the same type [illegible] but taking

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

for the fundamental curve, [illegible] equation of the surface

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z,$$

which represents an hyperbolic paraboloid.

10. A point on a [illegible] curve [illegible]

of the [illegible] and of the projection on the plane of x, y [illegible]. We may therefore represent a [illegible] curve by

$$z = g\left(\frac{y}{x}\right),$$
$$t = h\left(\frac{y}{x}\right).$$

The first equation represents a [illegible] having the axis of z as an axis, while the second represents a cylinder having its generators parallel to the axis of z. The equation

$$t = h\left(\frac{y}{x}\right)$$

may be called the type equation. If for the [illegible] curve we take the circle

$$x^2 + y^2 = r^2,$$

the equations of the [illegible] curve become

$$z = g\left(\frac{y}{x}\right),$$
$$t = h\left(\frac{y}{x}\right),$$

where [illegible] need not be written, as it is a constant.

The type equation

$$t = h\left(\frac{y}{x}\right),$$

may be considered as representing a curve in the plane of xy which is symmetrical with respect to the origin for on substituting for t its value, the type equation becomes

$$f(x,y) = \phi(\chi)$$

In the same way as in the case of surfaces, we observe that this may break up into a curve which is not symmetrical with respect to the origin together with the symmetric of this curve. Assuming that the points of any twisted curve cut out by planes through the axis of z can be represented by the same equation [illegible] the position of the cutting plane, [illegible] equations are

$$y = f(\chi)$$
$$z = f(\chi)$$

since χ can be taken as that parameter.

[illegible]

whatever whose points of section by planes through the axis of y are [illegible] [illegible] same [illegible] as in the foregoing [illegible]. In the case of a curve whose points of section are not symmetrical with respect to the axis of y, the same method would evidently be applicable, but would give in the first instance not only the proposed curve but also its symmetric with respect to the axis of y.

11. A twisted curve may also be represented by means of the equations

$$\varphi\left(z, \frac{r}{\rho}, \frac{r}{x}\right) = 0,$$

$$\psi(z, [illegible]) = 0$$

In the plane of ZOT these equations taken together will represent a certain number of points which are determined as the intersections of two plane curves. As the plane rotates about the axis of y these points [illegible]

As an example let us solve the following problem. The planes of three circles pass through the axis of z and each circle moves on three arbitrary [illegible] [illegible] by these circles, the surfaces traced out by [illegible] and the surface [illegible] by the circle which cuts these three circles orthogonally. Also determine the curves on which the centres of these three circles move and the curves traced out by [illegible]

[illegible]

$$s_1 = 0, \qquad s_2 = 0, \qquad s_3 = 0.$$

These circles will be referred to as the circles s_1, s_2, s_3 respectively. Suppose that the [illegible]

$$z^2 + t^2 + Az + Bt + C = 0$$

[illegible] s_1, [illegible] A, B, C [illegible] be determined by the condition that [illegible]

[illegible]

$(z_1, t_1), (z_2, t_2), (z_3, t_3)$. If these three points lie on the curves

$$z = g_1\left(\tfrac{y}{x}\right), \quad t = h_1\left(\tfrac{y}{x}\right),$$

$$z = g_2\left(\tfrac{y}{x}\right), \quad t = h_2\left(\tfrac{y}{x}\right),$$

$$z = g_3\left(\tfrac{y}{x}\right), \quad t = h_3\left(\tfrac{y}{x}\right),$$

respectively, the equation of the circle S_1 is

$$\begin{vmatrix} z^2+t^2 & z & t & 1 \\ g_1^2+h_1^2 & g_1 & h_1 & 1 \\ g_2^2+h_2^2 & g_2 & h_2 & 1 \\ g_3^2+h_3^2 & g_3 & h_3 & 1 \end{vmatrix} = 0,$$

which may be written in the form

$$z^2+t^2+2z\,D_1\left(\tfrac{y}{x}\right)+2t\,E_1\left(\tfrac{y}{x}\right)+F_1\left(\tfrac{y}{x}\right) = 0,$$

[illegible] of the [illegible] [illegible] form. The equations of the [illegible], therefore, become

(S_1) $$z^2+t^2+2z\,D_1\left(\tfrac{y}{x}\right)+2t\,E_1\left(\tfrac{y}{x}\right)+F_1\left(\tfrac{y}{x}\right) = 0,$$

(S_2) $$z^2+t^2+2z\,D_2\left(\tfrac{y}{x}\right)+2t\,E_2\left(\tfrac{y}{x}\right)+F_2\left(\tfrac{y}{x}\right) = 0,$$

(S_3) $$z^2+t^2+2z\,D_3\left(\tfrac{y}{x}\right)+2t\,E_3\left(\tfrac{y}{x}\right)+F_3\left(\tfrac{y}{x}\right) = 0.$$

The ordinary method for the determination

of a circle orthogonal to three given circles [illegible] applicable [illegible] for the equation of the circle

$$\begin{vmatrix} D_1\left(\frac{y}{x}\right)+z & E_1\left(\frac{y}{x}\right)+t & zD_1\left(\frac{y}{x}\right)+tE_1\left(\frac{y}{x}\right)+F_1\left(\frac{y}{x}\right) \\ D_2\left(\frac{y}{x}\right)+z & E_2\left(\frac{y}{x}\right)+t & zD_2\left(\frac{y}{x}\right)+tE_2\left(\frac{y}{x}\right)+F_2\left(\frac{y}{x}\right) \\ D_3\left(\frac{y}{x}\right)+z & E_3\left(\frac{y}{x}\right)+t & zD_3\left(\frac{y}{x}\right)+tE_3\left(\frac{y}{x}\right)+F_3\left(\frac{y}{x}\right) \end{vmatrix}=0.$$

This circle will be called the circle O. To obtain the equations of the surfaces traced out by these circles we substitute for t its value $\sqrt{x^2+y^2}$ in the equations of the circles and get for the surfaces

(S_1) $\quad z^2+x^2+y^2+2zD_1\left(\frac{y}{x}\right)+2E_1\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_1\left(\frac{y}{x}\right)=0,$

(S_2) $\quad z^2+x^2+y^2+2zD_2\left(\frac{y}{x}\right)+2E_2\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_2\left(\frac{y}{x}\right)=0,$

(S_3) $\quad z^2+x^2+y^2+2zD_3\left(\frac{y}{x}\right)+2E_3\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_3\left(\frac{y}{x}\right)=0,$

$$(O)\quad \begin{vmatrix} D_1\left(\frac{y}{x}\right)+z & E_1\left(\frac{y}{x}\right)+\sqrt{x^2+y^2} & zD_1\left(\frac{y}{x}\right)+E_1\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_1\left(\frac{y}{x}\right) \\ D_2\left(\frac{y}{x}\right)+z & E_2\left(\frac{y}{x}\right)+\sqrt{x^2+y^2} & zD_2\left(\frac{y}{x}\right)+E_2\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_2\left(\frac{y}{x}\right) \\ D_3\left(\frac{y}{x}\right)+z & E_3\left(\frac{y}{x}\right)+\sqrt{x^2+y^2} & zD_3\left(\frac{y}{x}\right)+E_3\left(\frac{y}{x}\right)\sqrt{x^2+y^2}+F_3\left(\frac{y}{x}\right) \end{vmatrix}=0.$$

For the surfaces [illegible] by [illegible] of the [illegible] we get

$(s_1 s_2)$ $\quad 2y\left[D_1\left(\frac{y}{x}\right) - D_2\left(\frac{y}{x}\right)\right] + 2(x^2+y^2)\left[E_1\left(\frac{y}{x}\right) - E_2\left(\frac{y}{x}\right)\right] + F_1\left(\frac{y}{x}\right) - F_2\left(\frac{y}{x}\right) = 0,$

$(s_2 s_3)$ $\quad 2y\left[D_2\left(\frac{y}{x}\right) - D_3\left(\frac{y}{x}\right)\right] + 2(x^2+y^2)\left[E_2\left(\frac{y}{x}\right) - E_3\left(\frac{y}{x}\right)\right] + F_2\left(\frac{y}{x}\right) - F_3\left(\frac{y}{x}\right) = 0,$

$(s_3 s_1)$ $\quad 2y\left[D_3\left(\frac{y}{x}\right) - D_1\left(\frac{y}{x}\right)\right] + 2(x^2+y^2)\left[E_3\left(\frac{y}{x}\right) - E_1\left(\frac{y}{x}\right)\right] + F_3\left(\frac{y}{x}\right) - F_1\left(\frac{y}{x}\right) = 0.$

[illegible]

common namely, the locus of the radical centre of the three circles. Writing the equations of the three circles in the forms,

(s_1) $\qquad (y + D_1)^2 + (t + E_1)^2 = D_1^2 + E_1^2 - F_1,$

(s_2) $\qquad (y + D_2)^2 + (t + E_2)^2 = D_2^2 + E_2^2 - F_2,$

(s_3) $\qquad (y + D_3)^2 + (t + E_3)^2 = D_3^2 + E_3^2 - F_3,$

[illegible]

(s_1) $\qquad y = -D_1\left(\frac{y}{x}\right), \quad t = -E_1\left(\frac{y}{x}\right),$

(s_2) $\qquad y = -D_2\left(\frac{y}{x}\right), \quad t = -E_2\left(\frac{y}{x}\right),$

(s_3) $\qquad y = -D_3\left(\frac{y}{x}\right), \quad t = -E_3\left(\frac{y}{x}\right)$

The curve traced out by their radical centre may be found by writing the equation of the circle O in the same form and getting the coordinates of the centre, or it may be found directly from the [illegible] at the bottom of page [illegible].

The equations are

$$x = \frac{\begin{vmatrix} F_1\left(\frac{y}{x}\right) - F_2\left(\frac{y}{x}\right) & E_1\left(\frac{y}{x}\right) - E_2\left(\frac{y}{x}\right) \\ F_2\left(\frac{y}{x}\right) - F_3\left(\frac{y}{x}\right) & E_2\left(\frac{y}{x}\right) - E_3\left(\frac{y}{x}\right) \end{vmatrix}}{\begin{vmatrix} D_1\left(\frac{y}{x}\right) - D_2\left(\frac{y}{x}\right) & E_1\left(\frac{y}{x}\right) - E_2\left(\frac{y}{x}\right) \\ D_2\left(\frac{y}{x}\right) - D_3\left(\frac{y}{x}\right) & E_2\left(\frac{y}{x}\right) - E_3\left(\frac{y}{x}\right) \end{vmatrix}},$$

$$y = \frac{\begin{vmatrix} D_1\left(\frac{y}{x}\right) - D_2\left(\frac{y}{x}\right) & F_2\left(\frac{y}{x}\right) - F_1\left(\frac{y}{x}\right) \\ D_2\left(\frac{y}{x}\right) - D_3\left(\frac{y}{x}\right) & F_3\left(\frac{y}{x}\right) - F_2\left(\frac{y}{x}\right) \end{vmatrix}}{\begin{vmatrix} D_1\left(\frac{y}{x}\right) - D_2\left(\frac{y}{x}\right) & E_1\left(\frac{y}{x}\right) - E_2\left(\frac{y}{x}\right) \\ D_2\left(\frac{y}{x}\right) - D_3\left(\frac{y}{x}\right) & E_2\left(\frac{y}{x}\right) - E_3\left(\frac{y}{x}\right) \end{vmatrix}}.$$

As another example let us determine the curve generated by the point of intersection of the [illegible]

$$y^2(a-x) - x^3 = 0,$$

with the straight line

$$x + y = a.$$

a is the distance OA and is the diameter of the circle OHA which lies in the plane ZOT. The cissoid is asymptotic to the line AR that is to the line

$$r = a.$$

[illegible]

$$r = \frac{x}{\sqrt{x^2+y^2}},$$

in the plane of xy (r, θ being polar coordinates in this plane). The intersections of the cissoid with the line OR are given by

$$y^2\left(\frac{x}{\sqrt{x^2+y^2}} - r\right) - r^3 = 0,$$

$$r - y = 0,$$

whence the twisted curve described by these intersections is

$$y^2\left[x \pm (x^2+y^2)\right] \pm (x^2+y^2)^2 = 0,$$

which breaks up into the equations

[illegible]

$$z^2[x - x^2 + y^2]\ [illegible]\ (x^2 + y^2)^2 = 0,$$

$$x^2 + y^2 - z^2 = 0$$

[illegible]

$$z^2[x + x^2 + y^2] + (x^2 + y^2)^2 = 0,$$

$$x^2 + y^2 - z^2 = 0.$$

It is readily seen that the points [illegible] the curve generated from the [illegible] is defined by the equations

$$x - z = 0,$$

$$- \frac{x}{x^2 + y^2}\ [illegible]$$

The equations of the curve [illegible] by [illegible] are

$$x^2 + y^2 - z^2 = 0,$$

$$(x^2 + y^2)\ [illegible] = 0.$$

The other equations

$$z^2(u - t) - t^3 = 0$$

become for the cissoid

$$z^2\left(\frac{x}{x^2 + y^2} - t\right) - t^3 = 0,$$

and the curve

$$(z^2 + t^2)t = 0.$$

Since the [illegible] equation

$$f - z = 0,$$

represents the lines OR and OR', the intersections of these two lines with the curve

$$y^2\left(\frac{x}{\sqrt{x^2+y^2}} - t\right) - t^3 = 0,$$
$$(x^2+y^2)\,t = a,$$

are the points of the [illegible] curve which lie in the plane OT. These intersections consist of the points L, L' and the origin counted six times. The curve is therefore of the eighth order and has the origin for a sextuple point. The curve consists of two branches traced out by the points L, L' which are symmetrical with respect to the planes of xy and xz. Now let us consider the curve traced out by the points R and R'.

The asymptote of the cissoid [illegible]

$$y^2\left(\frac{x}{\sqrt{x^2+y^2}} - t\right) - t^3 = 0,$$

[illegible]

which the corresponding [illegible] for this case

$$(y^2 + t^2)\,t^2 = 0,$$

is $$y = 0.$$

The intersections of the plane with the lines OR and OR' consist of the points R, R' and the origin counted twice. The curve therefore is of the fourth order, and has a double point at the origin. As the plane ZOT rotates from the position ZOX to the position ZOY, the asymptote RR' approaches OZ and in the limit coincides with OZ. In this motion the points R, R' trace out branches of the curve which in the limiting position coincide at O where the [illegible] meet. These points always lie on the lines OR, OR' and in the limit these lines coincide with the bisectors of the axes of x and y. These bisectors are therefore the tangents to the curve at the double point.

By rotating the plane from the position ZOX to the position ZOY' the abscissa of the curve is described.

[illegible]. Find the equation of the surface generated by the motion of a straight line which moves [illegible] upon two arbitrary curves.

The type equation of the straight line may be written in the form

$$\frac{y-y_0}{t-t_0} = \frac{y_1-y_0}{t_1-t_0}.$$

Suppose that the point (y_0, t_0) describes the curve

$$y = g_0\left(\frac{u}{x}\right), \quad t = h_0\left(\frac{u}{x}\right),$$

while the point (y_1, t_1) describes the curve

$$y = g_1\left(\frac{u}{x}\right), \quad t = h_1\left(\frac{u}{x}\right).$$

We must therefore substitute in the equation of the straight line

$$y_0 = g_0\left(\frac{u}{x}\right), \quad t_0 = h_0\left(\frac{u}{x}\right),$$

$$y_1 = g_1\left(\frac{u}{x}\right), \quad t_1 = h_1\left(\frac{u}{x}\right),$$

which then assumes the form

$$\frac{z - g_0\left(\frac{y}{x}\right)}{t - h_0\left(\frac{y}{x}\right)} = \frac{g_1\left(\frac{y}{x}\right) - g_0\left(\frac{y}{x}\right)}{h_1\left(\frac{y}{x}\right) - h_0\left(\frac{y}{x}\right)} = \Phi\left(\frac{y}{x}\right) \text{ say.}$$

Clearing of fractions we have after substituting for t its value (x^2+y^2),

$$z = \Phi\left(\frac{y}{x}\right)(x^2+y^2) - g_0\left(\frac{y}{x}\right) + [illegible]$$

which can be put in the form

$$z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right),$$

or in the equivalent form

$$z = y\phi_1\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right).$$

Either of the two forms where ϕ, ϕ_1, ψ are arbitrary is the equation of the [illegible].

3. The problem may also be [illegible], find the equation of a surface generated by a straight line which meets the axis of z, moves on an arbitrary curve and makes an angle varying according to a given law with the axis of z.

Here the type equation takes the form

$$z - z_0 = [illegible]$$

[illegible] taking

$$z_0 = \varphi_0\left(\frac{y}{x}\right),$$

$$t_0 = h_0\left(\frac{y}{x}\right),$$

$$\lambda = f\left(\frac{y}{x}\right),$$

we get for the equation of the surface

$$z = \lambda\,\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$$

If the [illegible] element is

$$z - z_0 = \lambda\, t,$$

as here

$$z_0 = c, \quad \lambda = f_1\left(\frac{y}{x}\right),$$

the surface is generated by a straight line which passes through the fixed point c on the axis of z and makes an angle varying according to a given law with the axis. The surface is therefore a cone whose equation is

$$z = \varphi_1\left(\frac{y}{x}\right)\sqrt{x^2+y^2} + c,$$

which may be written

$$z = [illegible]$$

If the vertex of the cone is at the origin the equation reduces to

$$z = [illegible]$$

If the type equation of the form is

$$p - p_0 = 0,$$

where $p = [illegible]$,

the surface is generated by the motion of a straight line which passes through the axis of z and remains parallel to the plane of xy. The surface is therefore a right conoid whose equation is

$$z = \varphi\left(\frac{y}{x}\right).$$

If the type equation is

$$t - t_0 = 0,$$

where $t = [illegible]$,

the surface is generated by the motion of a line which remains parallel to the axis of y. The surface is a right cylinder whose equation

$$\sqrt{x^2+y^2} = \varphi\left(\frac{y}{x}\right),$$

which may be written

$$r = \varphi\left(\frac{y}{x}\right).$$

Now let us write the type equation in the form

$$t - \psi\left(\frac{y}{x}\right) = \Phi\left(\frac{y}{x}\right)\left[z - \varphi\left(\frac{y}{x}\right)\right].$$

This means that the line passes through the point $\psi\left(\frac{y}{x}\right), \varphi\left(\frac{y}{x}\right)$ and makes an angle whose tangent is $\Phi\left(\frac{y}{x}\right)$ with the axis of z. If φ, ψ, Φ are uniform functions the type equation will represent two lines in the plane zOT which are symmetrical with respect to the axis of z. Let γ denote the angle which they make with this axis.

As an example suppose that the line passes through the circle

$$x^2+y^2=R^2,$$

and makes an angle with the [illegible] equal to that made by the tangent to the circle at the point where the generating line meets the circle. We have

$$\theta\left(\tfrac{x}{z}\right) = 0, \quad \psi\left(\tfrac{y}{x}\right) = R, \quad \phi(x) = -\tfrac{x}{y},$$

whence the equation of the surface is

$$\sqrt{x^2+y^2} - R = -\frac{x}{y}\, z,$$

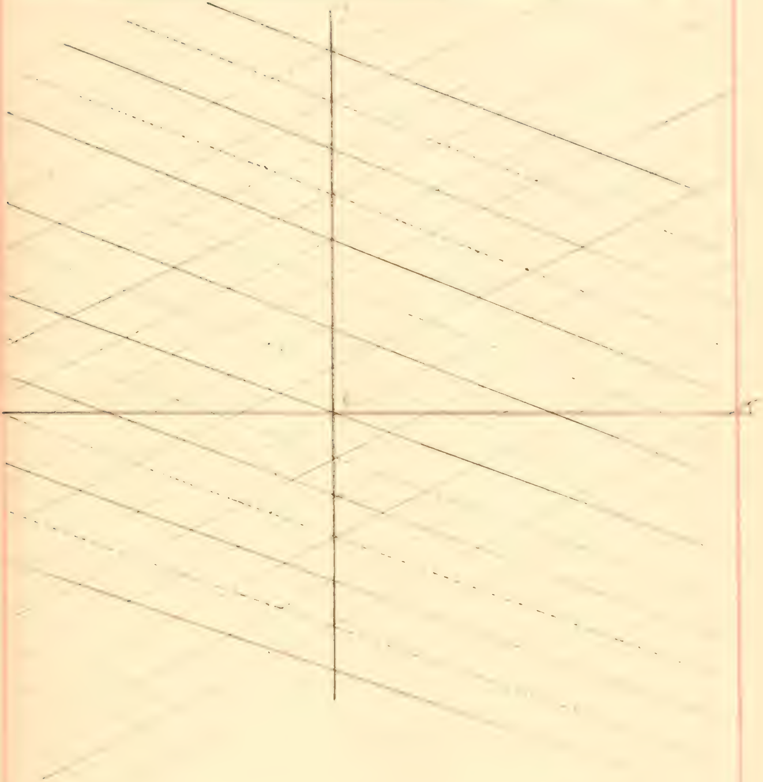
or $\quad (x^2+y^2)\,y^2 = (R\,y - x z)^2$ [illegible]

which represents a [illegible] surface. As an example of the case where ϕ, ψ, θ are not all uniform let us put

$$t = \psi\left(\tfrac{x}{y}\right) = R, \quad z = \phi(\tfrac{x}{y}) = [illegible]$$

These equations [illegible]

[illegible]



Each generator and each one of them both and the whole generation are symmetrical with respect to the axis of y. The generation

of the surface is

$$\sqrt{x^2+y^2} = h \cdot f(k)\left[z - \tan^{-1}\frac{y}{x}\right].$$

The sections of the surface by any plane through the axis of z divides the plane up into equal rhombuses. For the section consists of two sets of lines equally inclined with respect to the axis of z, and each set consists of equidistant parallel lines. In the figure the full lines are due to the generator moving one way, while the dotted lines are due to the generator moving on the other side. If the value of γ is 45° for any section that is if $f(k) = 1$, the section of the surface by any plane through the axis of z divides the plane up into equal squares and the size of the squares does not vary with the cutting plane. In this case the equation of the surface is

$$z - \tan^{-1}\frac{y}{x} = \sqrt{x^2+y^2} + \ldots$$

If $\Phi\left(\frac{z}{x}\right) = 0$, (that is if the generating line is parallel to the plane of xy) the two equations of lines coincide and the surface becomes the following

$$y = \Phi^{-1}\frac{z}{x}.$$

15. Find the general equation of surfaces generated by the motion of a circle whose plane always passes through the axis of y, the centre moving in one arbitrary curve and the circumference passing through another.

Let y_1, r_1 be the coordinates of the centre, $y_2\ r_2$ the coordinates of a point on the circumference. The equation of the circle will be

$$(y-y_1)^2 + (r-r_1)^2 = (y_2-y_1)^2 + (r_2-r_1)^2.$$

Suppose that

$$y_1 = \varphi_1\left(\frac{z}{x}\right), \quad r_1 = \psi_1\left(\frac{z}{x}\right),$$

on the curve described by the centre, and that $y_2 = \varphi_2\left(\frac{z}{x}\right), \quad r_2 = \psi_2\left(\frac{z}{x}\right),$

is the curve through which the circle [illegible]

The equation of the circle now becomes

$$(z-\varphi_1)^2+(t-\psi_1)^2=(\varphi_2-\psi_1)^2+(\varphi_2-\varphi_1)^2.$$

Substituting for t its value $\sqrt{x^2+y^2}$ in the equation of the circle we get for the equation of the surface

$$(z-\varphi_1)^2+(\sqrt{x^2+y^2}-\psi_1)^2=(\varphi_2-\psi_1)^2+(\varphi_2-\varphi_1)^2,$$

which may be put in the form

$$z^2+z\,f_1(x)+x^2+y^2+x\,f_2(x)+f_3(x)=0$$

or in the equivalent form

$$z^2+z\,f_1(x)+x^2+y^2+y\,f_2(x)+f_3(x)=0.$$

We shall of course obtain an equation of the same form if we allow the circle to move on three arbitrary curves. If we write the equation of the circle in the form

$$(z-\varphi_1)^2+(t-\psi_1)^2=[\theta_1(z)]^2,$$

which means that the centre moves on a given curve while the radius varies according to a given law. If θ_1 is a constant the surface is generated by a [illegible]

If ϕ_1 is constant the centre moves in a plane curve. If ψ_1 is constant the centre moves on a curve which lies on a right circular cylinder. If

$$\theta_1\left(\frac{y}{x}\right) = R,$$

$$\phi_1\left(\frac{y}{x}\right) = 0,$$

$$\psi_1\left(\frac{y}{x}\right) = c,$$

the surface is the anchor ring. If ψ_1 is zero, the centre moves on the axis of z. In this case the equation of the surface becomes

$$z^2 + z\,\phi_1\left(\frac{y}{x}\right) + x^2 + y^2 + \phi_3\left(\frac{y}{x}\right) = 0.$$

As an example suppose that the type equation of the variable circle is

$$z^2 + r^2 = \rho^2\left(\frac{y}{x}\right)^2,$$

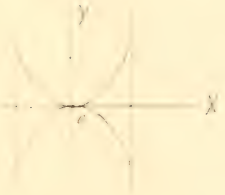
which shows that the centre is at the origin and that the radius is [illegible] times the tangent of the angle made with the axis of x by the corresponding radius OT. The equation of the surface is

[illegible]

The section made by the plane of xy is

$$y^2 = \frac{x^4}{r^2 - x^2}.$$

This is the section of this curve, the origin being the pole and the axis of the system of parallel circles. In this fact these circles are the projections of the circles on the plane xy.



The curve is represented by the figure. The origin is a point of osculation, the axis of X being a common tangent to the two branches. The lines

$$x = \pm r$$

are asymptotes of the curve. Suppose that the circle has a constant radius and that its centre moves on an ellipse in the plane of xy. Find the equation of the surface.

generated by the motion of this circle.
We have for the equation of the circle

$$z^2 + (r - r_1)^2 = c^2;$$

where r_1 is the radius vector of the ellipse, the center of the ellipse being the origin.
The polar equation of the ellipse is

$$r_1^2 = \frac{b^2}{1 - e^2\cos^2\theta}$$

where $\qquad \cos^2\theta = \frac{x^2}{x^2+y^2}$

whence $\qquad r_1^2 = \frac{b^2(x^2+y^2)}{x^2(1-e^2)+y^2},$

so that

$$r_1 = \sqrt{\frac{b^2(x^2+y^2)}{x^2(1-e^2)+y^2}}.$$

Substituting for r and r_1 their values in the equation of the circle we get for the equation of the anchor ring

$$z^2 + \left(\sqrt{x^2+y^2} - \sqrt{\frac{b^2(x^2+y^2)}{x^2(1-e^2)+y^2}}\right)^2 = c^2.$$

If the circle is tangent to the axis of z its equations is

$$r^2 + (z - z_1)^2 = 2cr.$$

Writing $\quad z_1 = \phi_1\left(\frac{y}{x}\right), \ c = \phi_2\left(\frac{y}{x}\right),$

the equation of the surface is

[illegible]

As an example suppose that an invariable [illegible]

with its centre on a helix.

[illegible]

$$t_1 = v,$$

$$z_1 = m \tan^{-1} \frac{y}{x},$$

[illegible]

$$\left[\left(z - m \tan^{-1} \frac{y}{x}\right)^2 + x^2 + y^2\right]^2 = 4c^2(x^2 + y^2).$$

As another example suppose that the [illegible]

the axis of z while its centre moves on the [illegible]

[illegible]

[illegible]

[illegible]

which joins the point [illegible]

the spiral takes the form

$$y^2 + \left(z - a\tan^{-1}\frac{y}{x}\right)^2 = c^2.$$

Substituting for c its value $\sqrt{x^2+y^2}$ we get for the equation of the surface

$$\left[x^2+y^2+\left(z-a\tan^{-1}\frac{y}{x}\right)^2 - c^2\right]^2 = 4(x^2+y^2)\left(z-a\tan^{-1}\frac{y}{x}\right)^2$$

The section of the surface by the plane of xy is

$$\left[x^2+y^2+a^2\left(\tan^{-1}\frac{y}{x}\right)^2 - c^2\right]^2 = 4a^2(x^2+y^2)\left(\tan^{-1}\frac{y}{x}\right)^2$$

or returning to polar coordinates

[illegible]

which breaks up into the four spirals

$$\rho + r = a\theta,$$

$$\rho - r = a\theta,$$

$$\rho + r = -a\theta,$$

$$\rho - r = -a\theta.$$

14. The central conic

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

will be determined if it is subjected to the conditions of passing through [illegible]

[illegible] of these [illegible] [illegible]

$$x = \varphi_1\left(\frac{w}{\chi}\right),$$
$$y = \psi_1\left(\frac{w}{\chi}\right);$$
$$x = \varphi_2\left(\frac{w}{\chi}\right),$$
$$y = \psi_2\left(\frac{w}{\chi}\right)$$

consecutively the 2 positions of the same [illegible]

$$\begin{vmatrix} x^2 & y^2 & 1 \\ \varphi_1 & \psi_1 & 1 \\ \varphi_2 & \psi_2 & 1 \end{vmatrix} = 0;$$

[illegible] that of the surface generated by the motion of this conic is

$$x^2(\psi_1^2 - \psi_2^2) + \chi^2\left(1 + \frac{w^2}{\chi^2}\right)(\varphi_2^2 - \varphi_1^2) = \varphi_2^2\psi_1^2 - \varphi_1^2\psi_2^2,$$

which may be written

$$x^2\left[\frac{\psi_1^2 - \psi_2^2}{[illegible]}\right] + \chi^2\left[\frac{[illegible]}{[illegible]}\right] = 1$$

[illegible] that the surface may be [illegible]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose equation may be written

$$x^2 + \chi^2\left(\frac{1}{b^2} + \frac{w^2}{[illegible]}\right) = 1.$$

it is necessary and sufficient that

$$\frac{\psi_1^2 - \psi_2^2}{\varphi_2^2\psi_1^2 - \varphi_1^2\psi_2^2} = c^2 \text{ [illegible]}$$

$$\frac{\varphi_2^2 - \varphi_1^2}{\varphi_2^2\psi_1^2 - \varphi_1^2\psi_2^2} = \frac{a^2 \text{ [illegible] } z^2}{x^2 + y^2}.$$

[illegible] is then the equation of the variable conic is

$$\begin{vmatrix} y^2 & z^2 & 1 \\ \text{[illegible]} & \text{[illegible]} & 1 \\ \varphi_2^2\left(\frac{y}{x}\right) & \psi_2^2\left(\frac{y}{x}\right) & 1 \end{vmatrix} = 0.$$

If the two guiding curves are [illegible] the surface will be algebraic. If they are transcendental the surface will in general be transcendental.

If [illegible]

$$\frac{\psi_1^2 - \psi_2^2}{\varphi_2^2\psi_1^2 - \varphi_1^2\psi_2^2} \equiv F\left(\frac{y}{x}\right),$$

$$\frac{\varphi_2^2 - \varphi_1^2}{\varphi_2^2\psi_1^2 - \varphi_1^2\psi_2^2} = G\left(\frac{y}{x}\right),$$

$F\left(\frac{x}{X}\right)$ and $G\left(\frac{x}{X}\right)$ be any algebraic functions.

15. In general five points determine a conic so we may predict a surface by allowing a conic whose plane passes through the axis of y to move on five arbitrary curves. Suppose that the equation of the conic is

$$at^2+2hty+by^2+2gt+2fy+c=0,$$

and that it moves on the five curves

$$t=\psi_1\left(\frac{x}{X}\right), \quad y=\theta_1\left(\frac{x}{X}\right);$$
$$t=\psi_2\left(\frac{x}{X}\right), \quad y=\theta_2\left(\frac{x}{X}\right);$$
$$t=\psi_3\left(\frac{x}{X}\right), \quad y=\theta_3\left(\frac{x}{X}\right);$$
$$t=\psi_4\left(\frac{x}{X}\right), \quad y=\theta_4\left(\frac{x}{X}\right);$$
$$t=\psi_5\left(\frac{x}{X}\right), \quad y=\theta_5\left(\frac{x}{X}\right).$$

Consider the conic in any one of its positions, and let the corresponding points for the five curves be (t_1,y_1), (t_2,y_2), (t_3,y_3), ... (t_5,y_5) respectively.

The [illegible]
then [illegible]

$$\begin{vmatrix} t^2 & ty & y^2 & t & y & 1 \\ t_1^2 & t_1y_1 & y_1^2 & t_1 & y_1 & 1 \\ t_2^2 & t_2y_2 & y_2^2 & t_2 & y_2 & 1 \\ t_3^2 & t_3y_3 & y_3^2 & t_3 & y_3 & 1 \\ t_4^2 & t_4y_4 & y_4^2 & t_4 & y_4 & 1 \\ t_5^2 & t_5y_5 & y_5^2 & t_5 & y_5 & 1 \end{vmatrix} = 0.$$

Substituting for [illegible] the equation of the generating conic becomes

$$\begin{vmatrix} t^2 & ty & y^2 & t & y & 1 \\ \psi_1^2(\tfrac{y}{x}) & \psi_1(\tfrac{y}{x})\theta_1(\tfrac{y}{x}) & \theta_1^2(\tfrac{y}{x}) & \psi_1(\tfrac{y}{x}) & \theta_1(\tfrac{y}{x}) & 1 \\ \psi_2^2(\tfrac{y}{x}) & \psi_2(\tfrac{y}{x})\theta_2(\tfrac{y}{x}) & \theta_2^2(\tfrac{y}{x}) & \psi_2(\tfrac{y}{x}) & \theta_2(\tfrac{y}{x}) & 1 \\ \psi_3^2(\tfrac{y}{x}) & \psi_3(\tfrac{y}{x})\theta_3(\tfrac{y}{x}) & \theta_3^2(\tfrac{y}{x}) & \psi_3(\tfrac{y}{x}) & \theta_3(\tfrac{y}{x}) & 1 \\ \psi_4^2(\tfrac{y}{x}) & \psi_4(\tfrac{y}{x})\theta_4(\tfrac{y}{x}) & \theta_4^2(\tfrac{y}{x}) & \psi_4(\tfrac{y}{x}) & \theta_4(\tfrac{y}{x}) & 1 \\ \psi_5^2(\tfrac{y}{x}) & \psi_5(\tfrac{y}{x})\theta_5(\tfrac{y}{x}) & \theta_5^2(\tfrac{y}{x}) & \psi_5(\tfrac{y}{x}) & \theta_5(\tfrac{y}{x}) & 1 \end{vmatrix} = 0$$

The coefficients [illegible] are [illegible] functions of $\frac{y}{x}$ and the equation takes the form

$$\Phi_1(\tfrac{y}{x})\,t^2 + \Phi_2(\tfrac{y}{x})\,ty + \Phi_3(\tfrac{y}{x})\,y^2 + \Phi_4(\tfrac{y}{x})\,t + \Phi_5(\tfrac{y}{x})\,y + \Phi_6(\tfrac{y}{x}) = 0.$$

Substituting for t its value [illegible]

we get for the equation of the surface

$$x^2\varphi_1\left(\tfrac{y}{x}\right) + xy\varphi_2\left(\tfrac{y}{x}\right) + y^2\varphi_3\left(\tfrac{y}{x}\right) + x\varphi_4\left(\tfrac{y}{x}\right) + y\varphi_5\left(\tfrac{y}{x}\right) + \varphi_6\left(\tfrac{y}{x}\right) = 0,$$

or the equivalent form

$$y^2\Psi_1\left(\tfrac{z}{x}\right) + yz\Psi_2\left(\tfrac{z}{x}\right) + z^2\Psi_3\left(\tfrac{z}{x}\right) + y\Psi_4\left(\tfrac{z}{x}\right) + z\Psi_5\left(\tfrac{z}{x}\right) + \Psi_6\left(\tfrac{z}{x}\right) = 0.$$

Either of these forms is the general equation of surfaces generated by the motion of a cone [illegible] passes through the origin. If the generating [illegible] [illegible] the surface will be [illegible]. If they are [illegible] the surface will be generated [illegible]. If, however,

$$\frac{\varphi_1(\;)}{\varphi_6(\;)} = [illegible],$$

$$\frac{\varphi_2(\;)}{\varphi_6(\;)} = [illegible];$$

$$\frac{\varphi_3(\;)}{\varphi_6(\;)} = [illegible],$$

$$\frac{\varphi_4(\;)}{\varphi_6(\;)} = [illegible],$$

$$\frac{\varphi_5(\;)}{\varphi_6(\;)} = [illegible],$$

where [illegible]

the surface will be algebraic if A, B, C, D, E are algebraic. Let us now see under what conditions the surface

$$x^2 q_1 + xz q_2 + z^2 q_3 + x q_4 + z q_5 + q_6 = 0,$$

[illegible]

[illegible] through the origin. The equation of this general surface [illegible]

$$a'x^2 + b'y^2 + c'z^2 + d'xy + e'xz + f'yz + g'x + h'y + i'z + 1 = 0,$$

[illegible]

$$x^2\left(a' + d'\frac{y}{x} + b'\frac{y^2}{x^2}\right) + zx\left(e' + f'\frac{y}{x}\right) + c'z^2 + x\left(g' + h'\frac{y}{x}\right) + i'z + 1$$

[illegible] represent a quadric surface, it is necessary [illegible]

[illegible]

[illegible]

16. [illegible] ...sted by a cubic whose plane passes through the axis of z [illegible]

$$x^3 q_1 + x^2 q_2 + x q_3 + x^2 z q_4 + z^2 x q_5 + z x q_6 + z q_7 + z^2 q_8 + z^3 q_9 + q_{10} = 0.$$

[illegible]

and see under what conditions this equation

[illegible]

$$ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fxy^2 + gzy^2 + hxz^2 + iyz^2 +$$
$$jxyz + kx^2 + ly^2 + mz^2 + nxy + pxz + qyz + rx + sy + tz + 1 = 0,$$

which may be written

$$x^3\left(a + d\frac{y}{x} + f\frac{y^2}{x^2} + b\frac{y^3}{x^3}\right) + x^2\left(k + n\frac{y}{x} + l\frac{y^2}{x^2}\right) + x\left(r + s\frac{y}{x}\right)$$
$$+ zx^2\left(e + j\frac{y}{x} + g\frac{y^2}{x^2}\right) + z^2x\left(h + i\frac{y}{x}\right) + zx\left(p + q\frac{y}{x}\right) + tz +$$
$$mz^2 + cz^3 + 1 = 0.$$

[illegible]

necessary and sufficient conditions are

$$\frac{q_1\left(\frac{y}{x}\right)}{q_{10}\left(\frac{y}{x}\right)} = a + d\frac{y}{x} + f\frac{y^2}{x^2} + k\frac{y^3}{x^3},$$

$$\frac{q_2\left(\frac{y}{x}\right)}{q_{10}\left(\frac{y}{x}\right)} = k + n\frac{y}{x} + l\frac{y^2}{x^2},$$

$$\frac{q_{3}\left(\frac{y}{x}\right)}{q_{10}\left(\frac{y}{x}\right)} = l + m\frac{y}{x},$$

[illegible]

$$\frac{q_5\left(\frac{y}{x}\right)}{q_{10}\left(\frac{y}{x}\right)} = h + i\frac{y}{x},$$

$$\frac{q_6\left(\frac{y}{x}\right)}{q_{10}\left(\frac{y}{x}\right)} = h + g\frac{y}{x},$$

[illegible]

[illegible]

[illegible]

17. [illegible]

order whose plane passes through the axis of z and which involves $\frac{n(n+1)}{2}$ [illegible]

6° Then if the equation of the curve is

$$(a_1, a_2, \ldots a_{\frac{n(n+1)}{2}})(x, z)^m = 0,$$

[illegible]

[illegible]

[illegible]

become a sphere.

[illegible] generated by the motion of a variable plane curve. Let the equation of the variable plane curve be

[illegible]

the coefficients of the equation being functions of x. If η, t' be the coordinates of the centre of curvature at the point y, t' of the curve, we have

$$\eta \ [illegible]$$

$$1+\left(\frac{dt}{dy}\right)^2+(t-t')\frac{d^2y}{dt^2}=0.$$

By means of the equation to the curve, y, $\frac{dt}{dy}$, $\frac{d^2t}{dy^2}$ can be found in terms of [illegible], hence from the other equations, we can by eliminating [illegible] obtain an equation between η and t', which is the equation to the evolute, [illegible] may denote the equation of the evolute by

$$[illegible]$$

the coefficients of the equation being functions of [illegible] for [illegible] in the equation of the [illegible] for the equation of the [illegible] by [illegible]

[illegible]

As an example let us find the equation of the surface generated by the evolute of the variable hypocycloid

$$(z)^{\frac{2}{3}} + (t)^{\frac{2}{3}} = \left[\varphi\left(\tfrac{y}{x}\right)\right]^{\frac{2}{3}}$$

We get for the equation of the evolute

$$(z+t)^{\frac{2}{3}} + (z-t)^{\frac{2}{3}} = 2\left[\varphi\left(\tfrac{y}{x}\right)\right]^{\frac{2}{3}},$$

whence that of the surface is

$$\left(z+\sqrt{x^2+y^2}\right)^{\frac{2}{3}} + \left(z-\sqrt{x^2+y^2}\right)^{\frac{2}{3}} = 2\left[\varphi\left(\tfrac{y}{x}\right)\right]^{\frac{2}{3}}.$$

19. Now let us consider the surface generated by the [illegible] in a plane [illegible] whose plane [illegible] through the axis of z, let

$$f(z, t) = 0,$$

be the equation of the curve, the coefficients of the equation being functions of $\frac{y}{x}$. Let z', t' be a point on this curve and let [illegible] be the [illegible] the [illegible] from this point. [illegible]

[illegible]

$$\sqrt{[illegible]}$$

[illegible]

y, z being the coordinates of a point on the [illegible]

and $f(y', z') = 0,$

the resultant

$$R(y, z) = 0,$$

[illegible]

for its value we get for the equation of [illegible]

$$R(y, \sqrt{x^2 + y^2}) = 0.$$

[illegible]

eighth degree, but reduces to the sixth in [illegible]

[illegible]

let us [illegible] the equation of the surface [illegible]

[illegible]

parabola

[illegible] $+ m z,$

$$m = \varphi\left(\frac{y}{x}\right).$$

We get for the equation of the parallel

[illegible]

$$-(r^2 - 4mz)^2\{r^2 + (z-m)^2\} = 0,$$

$$m = \varphi\left(\frac{y}{x}\right).$$

The equation of the surface is therefore

$$r^6 - \left[3(x^2+y^2) + z^2 + 8z\,\varphi\left(\tfrac{y}{x}\right) - 8\left[\varphi\left(\tfrac{y}{x}\right)\right]^2\right] r^4 +$$

$\Big[$ [illegible] $\Big] r^2$

$$-\left[x^2+y^2-4z\,\varphi\left(\tfrac{y}{x}\right)\right]^2\left[x^2+y^2+\left(z-\varphi\left(\tfrac{y}{x}\right)\right)^2\right] = 0.$$

19. We shall now give one more special case of the generation of surfaces.

Let us generate a surface by means of a lemniscate in the same way that an ellipsoid is generated by a variable ellipse. The variable ellipse lies in a plane through the axis of [illegible] while the extremity of the other moves on an ellipse whose centre is the origin [illegible]

of xy. To generate the corresponding surface from the lemniscate the [illegible] semi-axis of the lemniscate in the plane LOT must vary as the perpendicular on the plane of xy. Let the perpendicular on the plane of xy be

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

Transforming to polar coordinates, the equation becomes

$$\rho^2 = a^2 \cos 2\theta.$$

The equation of the generating lemniscate must then be

$$(\rho^2+z^2)^2 = a^2 \cos 2\theta\,(\rho^2-z^2),$$

which transformed to [illegible] becomes

$$(\rho^2+z^2)^2 = a^2 \frac{x^2-y^2}{x^2+y^2}(\rho^2-z^2).$$

Substituting the values of ρ^2 we get for the equation of the surface

$$(x^2+y^2+z^2)^2(x^2+y^2) = a^2(x^2-y^2)(x^2+y^2-z^2)$$

which represents a surface of the [illegible]

The singularity is a nodal line of the surface and just as an isolated point of a curve is a double point, so this line is a double line, and is cut by a plane in two coincident points. The surface having a quadruple point at the origin, [illegible] through the origin [illegible] generating [illegible] points [illegible] the [illegible] points. [illegible] lines through the origin [illegible] the surface [illegible] (there [illegible] of this [illegible]) are given by the equation

$$(x^2-y^2)(x^2+y^2-z^2)=0,$$

which is then [illegible] on the cone

$$x^2+y^2-z^2=0,$$

and [illegible]

$x - y^2 = 0$ [illegible]

hence the sections of the surface by any plane through the axis of z [illegible] and are isolated [illegible] the [illegible] of z; also the tangents to the [illegible] at the [illegible] at the angles between the axes of y and z their equation is

$$x^2 y^2 = 0,$$

whence the cone touched by these [illegible] tangents is $x^2 + y^2 z^2 = 0$.

These tangents therefore cut the surface in six coincident points at the origin as is easily accounted for. Each tangent cuts the [illegible] in four coincident points, for each branch of the [illegible] has a point of inflexion at the origin, also the intersection with the plane of yz consists of two points, whence the tangents to the [illegible] cut the surface in six coincident points.

It remains to show why the lines in the planes $x^2 - y^2 = 0$,

[illegible]

In each of these planes the generating lemniscate reduces to

$$(y^2 + z^2)^2 = 0,$$

hence every line through the origin in each of these planes [illegible]

the imaginary circle four times which with the intersection with the axis of z makes up the [illegible]

[illegible]

$$[illegible]$$

where γ_3 [illegible] γ_1, [illegible] γ_2, [illegible]

[illegible]

[illegible]

γ_2 and a branch commencing at γ_3 and extending

[illegible]

$\gamma_2 = \gamma_3$ and the [illegible]

$$t^2 = (z-z_1)(z-z_2)^2,$$

where z_2 is greater than z_1. The point z_2 has now closed up to z_3, the oval has joined the infinite branch and the point z_2 has become a double point. If, however, $z_2 = z_1$, then the equation becomes

$$t^2 = (z-z_1)^2(z-z_3),$$

and the oval has shrunk into the isolated point z_1. If we suppose

$$z_1 = z_2 = z_3,$$

the equation becomes

$$t^2 = (z-z_1)^3,$$

and the point z_1 becomes a cusp. If [illegible] remains invariable, the surface of revolution generated by the rotation [illegible] of the [illegible] [illegible] that the surface of revolution is the only curved surface whose plane sections through the axis of z can consist entirely of cubics

of these lines. Expand the cubic [illegible] in order that the surface may be a cubic we must have

$$z_1 + z_2 + z_3 = \text{constant},$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = \text{constant},$$

$$z_1 z_2 z_3 \quad = \text{constant},$$

that is these quantities are the roots of a cubic equation with constant coefficients; they are therefore constants and the generating cubic is immovable. Now let us suppose that the generating curve is the cubic

$$y^2 = (t - a_1)(t - a_2).$$

Suppose that t_1 moves on the [illegible] for instance

$$t = a \cos 2\theta,$$

while t_2 remains the circle

$$t = 2a,$$

We readily get for the equation of the surface

$$[illegible]$$

[illegible]

$$\left[(x^2+4axy+2ax^2+2ay^2)(x^2+y^2)^2+8a^2x^2y^2\right]^2=$$
$$(x^2+y^2)\left[(x^2+y^2)^3+4a^2x^2y^2+8a^2xy(x^2+y^2)\right]^2,$$

which represents a surface of the fourteenth order. The surface [illegible] [illegible] [illegible]

$$(x^2+y^2)^3 = [illegible]$$

20. The method may be readily extended to the [illegible] of n dimensions [illegible] [illegible] [illegible] the space of n dimensions will be obtained by eliminating [illegible]

[illegible]

and $\psi(x_n, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots \frac{x_{n-1}}{x_1}) = 0, \quad (n = 4)$

where the first equation represents a [illegible], and the second a plane curve lying in the plane $X_n O T$, the axes being supposed rectangular.

Biographical Sketch.

The author, Robert Armistead Sayre, was born in Montgomery, Alabama, August [illegible], 1866. His common school education was completed in 1883. In 1884 he entered the University of Alabama where he remained for two years. He then engaged in business pursuits until [illegible] when he entered the University of Virginia. In [illegible] he received the degree of [illegible] from the University of Alabama. In [illegible] he entered the Johns Hopkins University as candidate for the degree of Doctor of Philosophy, selecting Mathematics as his principal subject, and Astronomy as first and Physics as second subordinate. The author wishes to take this opportunity to express his gratitude to Professor Craig and to Professor Newcomb for their kindness to him during his connection with the Johns Hopkins University.